



Sieber, J. (2004). *Dynamics of delayed relay control systems with large delays*. <http://hdl.handle.net/1983/91>

Early version, also known as pre-print

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# DYNAMICS OF DELAYED RELAY CONTROL SYSTEMS WITH LARGE DELAYS

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**Abstract:** This paper investigates the dynamics of a piecewise linear delay differential equation modeling an inverted pendulum subject to delayed relay control. The inverted pendulum serves as an illustrative prototype example for an arbitrary saddle equilibrium. Delayed relay cannot give perfect stabilization of the equilibrium but generates small oscillations. On the other hand, one can construct simple switching manifolds that permit stable periodic orbits even with arbitrarily large delay in the control loop, provided the delay is known. Robustness of the stable periodic orbits with respect to parameter perturbations follows from their dynamical and structural stability in the dynamical systems sense.

**Keywords:** relay control, time delay, stabilizing feedback, periodic motion, dynamics

## 1. INTRODUCTION

Consider the problem of stabilizing a frictionless pendulum on a cart in its unstable upright position by force feedback to the cart, governed by the equation

$$\ddot{x} = \sin x + \cos x D \quad (1)$$

where  $x$  is the inclination angle of the pendulum. The feedback force  $D$  is applied to the cart to stabilize the origin  $x = 0$ . Since the uncontrolled inverted pendulum is a saddle without area contraction, stabilization by simple position feedback is impossible (ruling out control based on  $x$  only, see Atay (1999)). Furthermore, due to inherent delays,  $D$  is a function of the state some time  $\tau$  ago. Stépán (1989) showed that if  $D$  obeys a PD control law

$$D = -ax(t - \tau) - b\dot{x}(t - \tau) \quad (2)$$

then there exists a critical delay  $\tau_c = \sqrt{2}$  beyond which stabilization is no longer possible regardless of the choice of control gains. Moreover, the nonlinearities in (1) cannot be neglected as they cause complex small-amplitude dynamics for  $\tau$  close to  $\tau_c$  (see Sieber and Krauskopf (2003, 2004)). Similar results about

a maximal permissible delay apply to other linear control laws (Atay (1999)).

In order to overcome this restriction on  $\tau$ , one may choose a relay control law

$$D = -\varepsilon \operatorname{sign}[g(x(t - \tau), \dot{x}(t - \tau))] \quad (3)$$

where  $g$  is a linear, smooth, or piecewise affine function that divides  $\mathbb{R}^2$  into two simple domains  $\{g \geq 0\}$  and  $\{g < 0\}$ . Fridman et al. (2002) have demonstrated, using the prototype one-dimensional example

$$\dot{x} = \kappa x - \operatorname{sign}x(t - \tau), \quad (\kappa > 0) \quad (4)$$

that this type of control cannot achieve perfect stabilization but admits oscillations around 0. Moreover, they have shown that some of the corresponding periodic orbits are asymptotically orbitally stable if the delay  $\tau$  is smaller than  $\log 2 / \kappa$  in (4). In a generalization of the usual meaning of successful control, the control law (3) is called successful in stabilizing 0 if there exists a stable (symmetric) periodic orbit around 0. Note that, once a stable periodic orbit has been found, its amplitude can be made arbitrarily small by subsequently decreasing the  $\varepsilon$  in (3) step by step (see Fridman et al. (2003)).

In the consideration of these stable periodic orbits, the nonlinearities in (1) can be considered as a small perturbation (if  $\varepsilon \ll 1$ ) due to the discontinuity in (3). Since dynamically stable periodic orbits are also structurally stable, this small perturbation can be neglected in the search for stable periodic orbits. This permits one to rephrase the general problem of stabilization of the pendulum with delayed relay control as a piecewise linear problem:

*Problem 1.* Given  $\tau > 0$ , find a switching function  $g$  that divides  $\mathbb{R}^2$  into two simple domains  $G_+ = \{g(x, y) \geq 0\}$  and  $G_- = \{g(x, y) < 0\}$  such that symmetric stable periodic orbits exist in

$$\begin{aligned}\dot{x}(t) &= y(t) \\ \dot{y}(t) &= x(t) - \text{sign}[g(x(t-\tau), y(t-\tau))].\end{aligned}\quad (5)$$

It is important to note that  $g$  is permitted to depend on the delay  $\tau$  in (5). Problem 1 can be considered as a prototype for a general linear system of saddle type subject to a delayed discontinuous piecewise constant switch.

Section 2 discusses briefly how Problem 1 fits into the framework of previous and recent studies about the dynamics of piecewise smooth delay equations and how it relates to common solutions of delay problems in control theory. Section 3 considers linear switching functions  $g$  (and, hence, linear switching manifolds). It shows that, for linear  $g$ , there is a maximal permissible delay  $\tau$  where stable periodic orbits in system (5) exist. However, the permissible  $\tau$  is larger than the one proposed by Fridman et al. (2002, 2003). Section 4 constructs a function  $g$  that permits stable periodic orbits in (5) for any given delay  $\tau$  where depends  $g$  on  $\tau$ . Section 5 states how the results of Section 4 can be generalized and discusses open problems for future consideration.

## 2. BACKGROUND

Equation (5) fits into the framework of delayed relay dynamics, which was also studied by Fridman et al. (2002, 2003). It is a linear system with one unstable direction that is being controlled by a delayed relay with a single switching line. The results of Fridman et al. (2002, 2003) can be applied directly to system (5) for the special case  $g(x, y) = x + y$ . The first part of Section 3 will discuss this.

Shustin et al. (2003) have studied in detail a second-order system of the form  $\varepsilon \ddot{x} = -\dot{x} + F(x, t) - \text{sign}[x(t-\tau)]$ , which is an area contracting saddle subject to delayed relay control. In particular, singular perturbation techniques one allow to carry over the results for one-dimensional systems like (4) for small  $\varepsilon$ . See Shustin et al. (2003) for further references.

Bayer and an der Heyden (1998) have studied a neutral linear center subject to delayed relay control, that is,

$\ddot{x} = -x + \text{sign}[x(t-\tau) - \Theta]$ . The authors found a rich zoo of periodic orbits in the two-parameter space  $(\tau, \Theta)$ .

From a control theoretic point of view there are often restrictions on the choice of  $g$  such as dependence on some output variables only. However, for system (5) stabilization using control depending only on  $x$  is impossible. There are also further requirements on a good control scheme such as robustness for all delays smaller than the delay  $\tau$  the control was designed for. If the delay occurs mainly in the implementation of the control action (input delay) one can compensate for the delay completely. See Roh and Oh (1999) for a relay scheme that compensates the input delay and achieves perfect stabilization after time  $2\tau$ .

The focus of the research presented here is the dynamics of a piecewise linear (or piecewise smooth) delay differential equations, the existence of stable periodic orbits and mechanisms for their destruction or loss of stability some of which were shown for several simple examples in Holmberg (1991). In order to gain an understanding of these mechanisms it is beneficial to consider general piecewise affine switching curves. This effectively results in a hybrid control scheme rather than a classical relay scheme from a control theoretic point of view.

## 3. LINEAR SWITCHING FUNCTION

System (5) has the infinite-dimensional phase space  $C([-\tau, 0]; \mathbb{R}^2)$ . However, at any given moment  $t$  the head point of a solution of system (5) follows the flow of a two-dimensional ordinary differential equation (ODE) corresponding to (5) in either  $G_+$  or  $G_-$ . These flows can be computed analytically and expressed by the affine maps

$$S_{\pm}(t; v) = A(t)v \pm v_0(t)$$

where

$$A(t) = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}, \quad v_0(t) = \begin{bmatrix} 1 - \cosh(t) \\ -\sinh(t) \end{bmatrix}.$$

$S_+$  has the saddle fixed point  $(1, 0)$  with the backward attracting invariant line  $l_+^u = (1 + s, s)$  ( $s = \pm e^t \in \mathbb{R}$ ) and the forward attracting invariant line  $l_+^s = (1 - s, s)$ . Correspondingly,  $S_-$  has the saddle fixed point  $(0, -1)$  with the backward attracting invariant line  $l_-^u = (-1 + s, s)$  and the forward attracting invariant line  $l_-^s = (-1 - s, s)$ . The lines  $l_{\pm}^u$  and  $l_{\pm}^s$  form a square (see Fig. 1). All periodic solutions of (5) have to lie within this square.

Let

$$g(x, y) = x \cos \alpha + y \sin \alpha, \quad (\alpha \in (0, \pi/2)).$$

In the special case  $\alpha = \pi/4$ , the dynamics in the physical space  $\mathbb{R}^2$  of (5) can be split into equations for  $u_1 = x + y$  and  $u_2 = x - y$

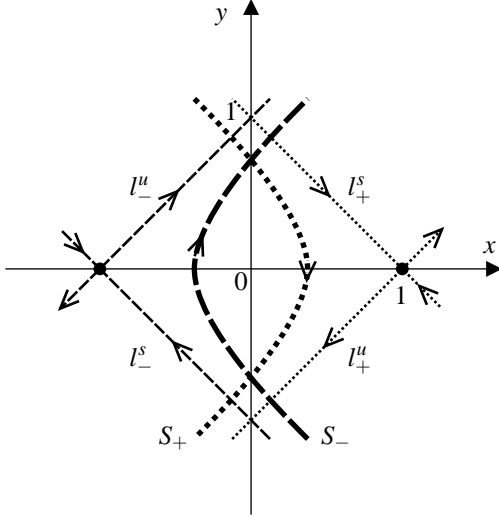


Fig. 1. The flows  $S_+$  (dotted) and  $S_-$  (dashed) and their invariant lines

$$\dot{u}_1(t) = u_1(t) - \text{sign}[u_1(t - \tau)] \quad (6)$$

$$\dot{u}_2(t) = -u_2(t) - \text{sign}[u_1(t - \tau)] \quad (7)$$

where (6) does not depend on (7). Fridman et al. (2002) have treated extensively the problem of existence and stability of periodic solutions for the one-dimensional case described by (6). Their results concerning (6) can be summarized as follows:

**Theorem 1.** (Fridman et al. (2002)). Let  $\tau_* < \log 2$ ,  $p = 2(\tau_* - \log(2 - e^{\tau_*}))$ , and  $k \in \mathbb{N}$ . Then there exists a periodic solution with period  $p$  of (6) with  $\tau = \tau_* + kp$ . If  $k = 0$ , this periodic solution is orbitally asymptotically stable. If  $k \geq 1$ , it is unstable.

Fridman et al. (2002) called these periodic solutions *k-frequency steady modes* where  $k + 1$  is the number of zeroes within the delay interval. Moreover, the authors observed that the zero-frequency steady modes attract nearby trajectories (in the  $C([-\tau, 0]; \mathbb{R})$  topology) in finite time.

Theorem 1 carries over exactly to system (6)–(7), and, hence, to system (5) with  $g(x, y) = (x + y)/\sqrt{2}$ . In system (5), the periodic solution switches in  $(x_-, y_-) = (0, e^{\tau_*} - 1)$  from  $S_-$  to  $S_+$ , and in  $(x_+, y_+) = (0, 1 - e^{\tau_*})$  from  $S_+$  to  $S_-$ . The results of Fridman et al. (2002) also imply that there exist no stable periodic orbits for delays  $\tau \geq \log 2$  in (5) if  $\alpha = \pi/4$ .

The fact that case  $\alpha = \pi/4$  is special can be understood geometrically since the undelayed switching line  $x + y = 0$  is only shifted by the flows  $S_+$  (downwards) and  $S_-$  (upwards) but not rotated (see Fig. 2). The critical value for  $\tau$  is determined by the condition that the intersection point of the line  $S_-(\tau, \{x + y = 0\})$  with the axis  $x = 0$  is  $(0, 1)$ . Then,  $S_-(\tau; \{x + y = 0\})$  coincides with the line  $l_+^s$  resulting in a heteroclinic connection.

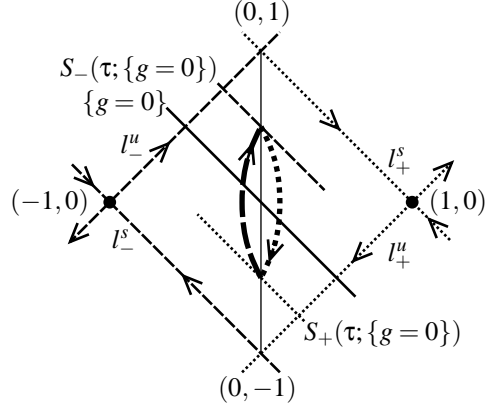


Fig. 2. The case  $\alpha = \pi/4$ . The switching line  $\{g = x + y = 0\}$  and its  $\tau$ -images under both flows are shown together with a periodic orbit for  $\tau < \log 2$ .

In the case  $\alpha \neq \pi/4$  the expanding and contracting directions of  $S_{\pm}$  can no longer be decoupled. Geometrically, this can be seen by the fact that the images of the undelayed switching line  $\{g(x, y) = x \cos \alpha + y \sin \alpha = 0\}$  rotate under the action of the flows  $S_{\pm}$ . There are now two conditions (8) and (9) on  $\alpha$  and  $\tau$  for the existence of stable periodic orbits.

**Theorem 2.** Let  $k \in \mathbb{N}$ ,

$$\tau_* < \log(1 + \tan \alpha), \quad (8)$$

and define

$$p = 2 \left[ \tau_* + \log \left( \frac{e^{\tau_*} \tan \alpha + 1 - e^{\tau_*}}{\tan \alpha + 1 - e^{\tau_*}} \right) \right].$$

Denote the intersection points of  $S_{\pm}(\tau_*; \{g = 0\})$  with the axis  $\{x = 0\}$  by  $P_{\pm}$ . Then there exists a periodic orbit of system (5) with  $\tau = \tau_* + kp$  that follows  $S_-$  from  $P_+$  to  $P_-$  and  $S_+$  from  $P_-$  to  $P_+$ . This orbit has period  $p$ . If  $k \geq 1$ , it is unstable. If  $k = 0$ , and, moreover,

$$\alpha \leq \pi/4, \quad \text{or} \quad \alpha > \pi/4 \quad \text{and} \quad \tau_* < \frac{1}{2} \left[ \log \left( \frac{\tan \alpha + 1}{\tan \alpha - 1} \right) \right], \quad (9)$$

the periodic orbit is orbitally asymptotically stable.

The proof of Theorem 2 is presented in the appendix. Note that condition (8) is equivalent to the condition that the moduli of the  $y$ -coordinates of the points

$$P_{\pm} = \left( 0, \mp \frac{\cosh \tau_* \cos \alpha - \sinh \tau_* \sin \alpha - \cos \alpha}{\sinh \tau_* \cos \alpha - \cosh \tau_* \sin \alpha} \right)$$

are less than 1 (see Fig. 3 and Fig. 4). The points  $P_{\pm}$  are mapped onto each other under  $S_{\pm}$ , respectively, that is,  $S_-(p/2; P_+) = P_-$  and  $S_+(p/2; P_-) = P_+$ . This gives rise to the periodic orbits asserted in Theorem 2. Furthermore, condition (9) is equivalent to the condition that the slope of  $S_{\pm}(\tau_*; \{g = 0\})$  is negative (Fig. 3), which implies stability of the periodic orbit of system (5) with  $\tau = \tau_*$  ( $k = 0$ ).

For  $\alpha$  close to  $\pi/2$ , condition (8) is satisfied even for large  $\tau_*$ . However, the periodic orbit of (5) with

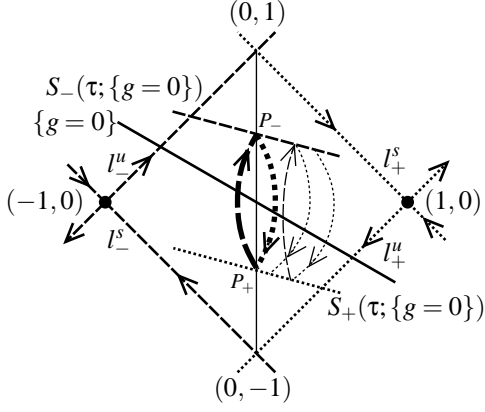


Fig. 3. The case  $\alpha > \pi/4$  when (8) and (9) hold. In addition to the stable periodic orbit another orbit is drawn to illustrate the dynamics of the return map between  $S_{\pm}(\tau; \{g=0\})$ .

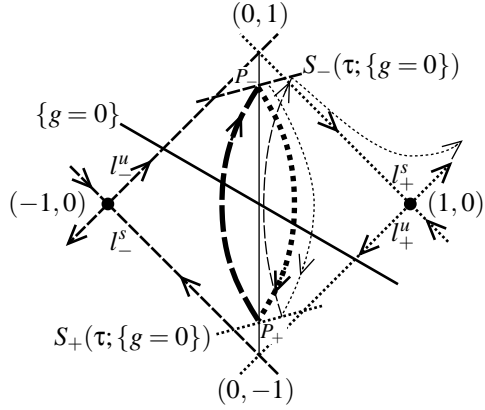


Fig. 4. The case  $\alpha > \pi/4$  when (8) holds but not (9). In addition to the unstable periodic orbit another orbit is drawn to illustrate the dynamics of the return map between  $S_{\pm}(\tau; \{g=0\})$ .

$\tau = \tau_*$  ( $k = 0$  in Theorem 2) undergoes a symmetry-breaking pitchfork bifurcation when the inequality for  $\tau_*$  in (9) turns into an equality. The bifurcation is degenerate. That is, all asymmetric periodic orbits exist at the same parameter value. After this bifurcation the symmetric periodic orbit is unstable (see Fig. 4).

If the delay  $\tau$  is such that the periodic orbit has more than one period per delay interval ( $k \geq 1$  in Theorem 2), then the periodic orbit is unstable. The conditions (8) and (9) allow for stable periodic orbits for  $\tau_*$  larger than  $\log 2$ . That is, for the optimal  $\alpha = \arctan \sqrt{2} \approx 0.304\pi$ , the conditions (8) and (9) allow for any  $\tau_* < \log(\sqrt{2} + 1)$ .

#### 4. NONLINEAR OR PIECEWISE AFFINE SWITCHING FUNCTION

Let  $\tau > \log 2$  be given. The goal of this section is to construct a piecewise affine function  $g$  dividing  $\mathbb{R}^2$  into two simple domains such that (5) has a stable periodic orbit. More precisely, it is sufficient to

construct two domains,  $G_+$  for the flow  $S_+$  and  $G_-$  for the flow  $S_-$ , and a piecewise affine boundary  $B$  separating them. Then, a piecewise affine function  $g$  can always be chosen such that  $\text{cl } G_+ = \{g(x, y) \geq 0\}$  and  $\text{cl } G_- = \{g(x, y) \leq 0\}$ . We choose the boundary  $B$  in the following way:

Let  $h \in (0, 1/2)$  be such that

$$e^\tau \in (h^{-1} - 1, h^{-1}). \quad (10)$$

Define the points  $b_+ = (0, -1 + 2h)$  and  $b_- = (0, 1 - 2h)$  and the direction vector  $w = (-1, 1)$  in  $\mathbb{R}^2$ , and choose  $B$  consisting of the 3 pieces of straight lines (see Fig. 5) as

$$\begin{aligned} B_+ &= \{b \in \mathbb{R}^2 : b = b_+ + tw, t < 0\}, \\ B_0 &= \{b \in \mathbb{R}^2 : b = b_+ + t(b_- - b_+), t \in [0, 1]\}, \\ B_- &= \{b \in \mathbb{R}^2 : b = b_- + tw, t > 0\}. \end{aligned}$$

The regions are chosen such that  $(1, 0) \in G_+$  and  $(-1, 0) \in G_-$ . The image of  $B_-$  under  $S_+(\tau; \cdot)$  inter-

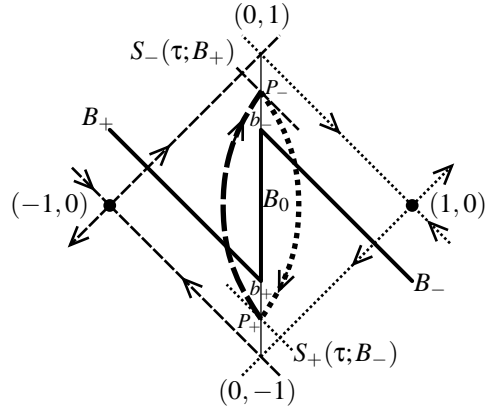


Fig. 5. Construction of a piecewise affine switching line  $B$  and the stable periodic orbit.

sects the axis  $\{x=0\}$  in  $P_+ = (0, 1 - 2e^\tau h)$ , which is located between  $(0, -1)$  and  $b_+$  due to condition (10). The image of  $B_+$  under  $S_-(\tau; \cdot)$  intersects  $\{x=0\}$  in  $P_- = (0, 2e^\tau h - 1)$ , which is located between  $b_-$  and  $(0, 1)$ . This gives rise to the existence of a symmetric periodic orbit  $W$  of period  $p = 4\tau - 2\log(1 - h^{-1})$  following  $S_+$  from  $P_-$  to  $P_+$  and following  $S_-$  from  $P_+$  to  $P_-$ .

**Lemma 3.** The periodic orbit  $W$  is orbitally asymptotically stable.

**Proof:** The orbit  $W$  intersects the boundary  $B$  twice transversally. Hence, any initial condition of system (5) that is sufficiently close to  $W$  in the  $C([-\tau, 0]; \mathbb{R}^2)$  topology will have a trajectory that follows  $S_+$  from  $S_-(\tau; B_+)$  to  $S_+(\tau; B_-)$  and follows  $S_-$  from  $S_+(\tau; B_-)$  to  $S_-(\tau; B_+)$  after time  $2\tau$ . Thus, asymptotic orbital stability of  $W$  is equivalent to the asymptotic stability of the fixed point  $P_+$  with respect to the return map  $\mathcal{R} : S_+(\tau; B_-) \rightarrow S_+(\tau; B_-)$  under  $S_+(p/2; \cdot) \circ S_-(p/2; \cdot)$ . The map  $\mathcal{R}$  is a linear contraction with rate  $e^{-p}$  since

$S_+(\tau; B_-)$  and  $S_-(\tau; B_+)$  are stable invariant leaves of  $S_+$  and  $S_-$ . Consequently, its fixed point  $P_+$  is asymptotically stable.  $\square$

*Remarks* The periodic orbit  $W$  is also structurally stable. That is, it is robust with respect to small nonlinearities such as in (1), or small perturbations of the parameters, for example  $\tau$ . However, this tolerance is exponentially small since (10) gives effectively a condition on  $\tau$  once  $h$  is chosen.

The convergence rate  $e^{-P}$  for the return map  $\mathcal{R}$  can be improved to quadratic convergence by choosing  $B_{\pm}$  such that  $S_+(\tau; B_-)$  is tangential to  $S_-$  in  $P_+$ . However, this induces a return map  $\mathcal{R}$  that is nonlinear.

The basic idea behind the construction of the boundary  $B$  (and, hence,  $g$ ) is that one can compensate for the delay  $\tau$  by negative hysteresis. Since the flows  $S_+$  and  $S_-$  have equilibria one can compensate for arbitrarily large delays by a finite amount of negative hysteresis. Instead of hysteresis one can exploit the additional stable dimension to construct a unique  $g$ . One could choose a smooth nonlinear pair of  $B$  and  $g$  instead of the piecewise affine  $B$  (and  $g$ ) by smoothing the corners at the points  $b_+$  and  $b_-$ .

If system (5) results from a linearization around an equilibrium of a nonlinear system (for example, system (1), (3)), the construction gives rise to a stable periodic orbit  $W_{\varepsilon}$  for every  $\varepsilon$  in (3). Once the system has settled to  $W_{\varepsilon_0}$  for a certain moderately small  $\varepsilon_0$ , one can subsequently decrease  $\varepsilon$  step by step, effectively continuing the stable periodic orbit in the parameter  $\varepsilon$ . In this way, one can obtain stable periodic orbits  $W_{\varepsilon}$  of arbitrarily small amplitude. This algorithm was described as  $\varepsilon$ -stabilization by Fridman et al. (2003).

Section 1 discussed already that in a nonlinear problem like (1), (3) the nonlinearities can be considered as small perturbations under which a stable periodic orbit as displayed in Fig. 5 will persist. However, if the nonlinearity is known, the construction of the boundary  $B$  can be improved by choosing the nonlinear stable fibers of the saddle for  $B_{\pm}$  instead of the straight lines in Fig. 5. This would allow one to use this construction for the fully nonlinear problem even for only moderately small  $\varepsilon$ .

## 5. CONCLUSIONS AND OUTLOOK

System (5) is a prototype for the more general situation of a saddle with one unstable direction and one or several stable directions. The construction of the switching function  $g$  for large delays  $\tau$  works for the general case in exactly the same manner as proposed in Section 4.

As has been pointed out in a remark in Section 4 already, the return maps composed by the returns of the single flows to their delayed switching lines can

become nonlinear even though all involved flows and switching lines are affine. This gives rise to classical bifurcation scenarios under parameter variations for the Poincaré maps near the periodic orbits constructed in Sections 3 and 4. Apart from these classical bifurcations, one can expect grazing events whenever the head point of a stable periodic orbit touches the undelayed switching line along its evolution. Phenomena like these have been studied by Holmberg (1991), and Bayer and an der Heyden (1998). In order to construct and control bifurcation scenarios without too much technical overhead it is advantageous to have the freedom to choose an arbitrary piecewise affine or nonlinear switching line. This freedom effectively gives us an extremely flexible method to introduce and vary parameters.

Furthermore, it is worth exploring how similar constructions as the one of Section 4 can be applied in other practically relevant situations, for example the stabilization of an unstable focus or a saddle periodic orbit in the presence of a large but known delay.

## 6. ACKNOWLEDGMENTS

The author thanks Bernd Krauskopf and Piotr Kowalczyk for helpful discussions and encouragement. This research is supported by EPSRC grant GR/R72020/01.

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## APPENDIX

Proof of Theorem 2: Denote the periodic orbit of period  $p$  of system (5) with delay  $\tau = \tau_* + kp$  by  $W$ .

*Instability of  $W$  for  $k \geq 1$ :* Let us consider only initial conditions in  $C([-\tau, 0]; \mathbb{R}^2)$  that follow always either  $S_+$  or  $S_-$ , stay close to  $W$  in their entire history interval, and switch between  $S_+$  and  $S_-$  only close to the switching events of  $W$ . The evolution of these initial conditions under system (5) is governed by the recursion

$$r_n = \tau + \rho(x_{n-2k}, y_{n-2k}) - r_{n-1} - \dots - r_{n-2k} \quad (11)$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = - \left( A(r_{n-1}) \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} + v_0(r_{n-1}) \right) \quad (12)$$

where  $r_n$  is the time difference between the subsequent switching moments  $t_{n+1}$  and  $t_n$  between the flows  $S_{\pm}$ , and  $(x_n, y_n) \in [-1, 1] \times [0, 1]$  is the location of the switch at the moment  $t_n$  if the switch is from  $S_-$  to  $S_+$ , and the negative of the location of the switch at  $t_n$  if the switch is from  $S_+$  to  $S_-$ . The function  $\rho(x, y)$ , defined by

$$\rho(x, y) = \log \left[ \frac{1 + \sqrt{1 + (\tan \alpha - 1)((x-1)^2 - y^2)}}{(1-x-y)(1+\tan \alpha)} \right],$$

measures the time it takes from  $(x, y)$  to the next crossing of the undelayed switching line  $\{g = 0\}$  under the the flow  $S_+$ . Relation (11) states that the switching moment is determined by the crossing of the  $\{g = 0\}$  line  $2k+1$  switches (that is, time  $\tau$ ) ago. Relation (12) states that the location of the switching event at  $t_n$  is the  $S_{\pm}(r_{n-1}; \cdot)$  image of the previous one. The reflection symmetry of the problem has been reduced in (12) by mapping the switching points close to  $P_+$  by  $-1$ . Thus,  $y_n > 0$  as long as the evolution stays close to  $W$ .

The recursion (11)–(12) defines a map of dimension  $6k$ . This map has the fixed point  $(x_*, y_*) = P_-$ ,  $r_* = p/2$  corresponding to  $W$ . Hence, for the proof of the orbital instability of  $W$  it is sufficient to prove that the linearization of the map defined by (11)–(12) in  $(x_*, y_*, r_*)$  has at least one eigenvalue of modulus greater than 1. The characteristic polynomial  $\chi$  of the linearization has order  $6k$  and is of the form

$$\chi(\lambda) = \lambda^{4k-2} \left( \sum_{j=0}^{2k+1} a_j \lambda^j \right)$$

where  $a_{2k+1} = 1$ ,  $a_{2k} = 1 + 2 \cosh r_*$ ,  $a_j = 2 + 2 \cosh r_*$  for  $j = 1, \dots, 2k-1$ , and

$$a_0 = (1 + \cosh r_*)^{-1} [1 + 3 \cosh r_* + 2 \cosh^2 r_* - \partial_2 \rho(x_*, y_*) (\cosh r_* + 1) - \partial_1 \rho(x_*, y_*) \sinh r_*].$$

The combination of the first two iterations of the recursive Schur-Cohn instability criterion provides a sufficient condition for the instability of the linear map corresponding to  $\chi$ :

$$|a_0| > 1 \quad \text{or} \quad |a_1 - a_0 a_{2k}| > |a_0^2 - 1|. \quad (13)$$

Fig. 6 shows the (numerically computed) boundaries for the instability condition (13) in the  $(\alpha, r_*)$ -plane showing that always at least one root of  $\chi$  is outside of the unit circle.

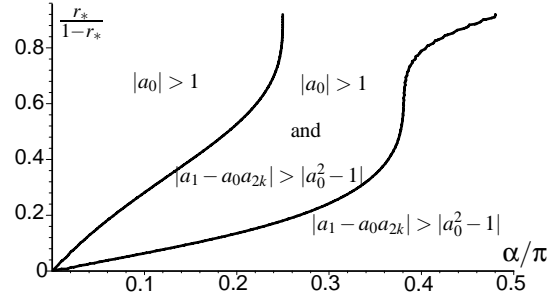


Fig. 6. Boundaries for the Schur-Cohn instability criterion (13). Note that there is a one-to-one correspondence between  $\tau$  and  $r_*$ :  $\tau = \tau_* + 2kr_*$ .

*Stability of  $W$  for  $k = 0$ :* The orbit  $W$  intersects the line  $B = \{g = 0\}$  twice transversally. Hence, any initial condition of system (5) that is sufficiently close to  $W$  in the  $C([-\tau, 0]; \mathbb{R}^2)$  topology will have a trajectory that follows  $S_+$  from  $B_- = S_-(\tau; B)$  to  $B_+ = S_+(\tau; B)$  and follows  $S_-$  from  $B_+$  to  $B_-$  after time  $2\tau$ . Thus, asymptotic orbital stability of  $W$  is equivalent to the asymptotic stability of the fixed point  $P_-$  with respect to the return map  $\mathcal{R}: B_- \rightarrow B_-$  defined by the  $S_+$  map from  $B_-$  to  $B_+$ , multiplied by  $-1$  (taking the reflection symmetry into account). Denote the  $y$ -coordinate of  $P_-$  by  $y_0$  and the slope of  $B_-$  by  $a$ . Then, points  $(x, y) \in B_-$  satisfy  $y = ax + y_0$ . Condition (8) is equivalent to  $y_0 \in (0, 1)$ . Condition (9) is equivalent to  $a < 0$ . The return map  $\mathcal{R}$  is governed by the implicit expression

$$(a\tilde{x} + y_0)^2 - (\tilde{x} + 1)^2 = (ax + y_0)^2 - (x - 1)^2 \quad (14)$$

where  $\tilde{x}$  is the  $x$ -coordinate of the image under  $\mathcal{R}$  of the point  $(x, ax + y_0) \in B_-$ . Linearization of (14) in  $x = \tilde{x} = 0$  implies that

$$\left. \frac{\partial \tilde{x}}{\partial x} \right|_{x=0} = \frac{y_0 a + 1}{y_0 a - 1}$$

which has modulus less than 1 if and only if  $a < 0$ .  $\square$